Olaf Lechtenfeld
Marcus Sperling

Fundamental Interactions

Tutorial #1
Thursday, April 14

Problem 1: Relativistic Kinematics In a two-body scattering event, $A + B \to C + D$, it is convenient to introduce the *Mandelstam variables*

$$s := \frac{(p_A + p_B)^2}{c^2} , \qquad t := \frac{(p_A - p_C)^2}{c^2} , \qquad u := \frac{(p_A - p_D)^2}{c^2} . \tag{1}$$

The theoretical virtue of the Mandelstam variables is that they are Lorentz invariants, with the same value in any inertial system. Experimentally, though, the more accessible parameters are energies and scattering angles.

a) Show that $s + t + u = m_A^2 + m_B^2 + m_C^2 + m_D^2$.

b) Find the centre-of-mass energy of A, in terms of s, t, u, and the masses.

c) Find the Lab (B at rest) energy of A.

d) Find the total centre-of-mass energy ($E_{tot} = E_A + E_B = E_C + E_D$).

Problem 2: Lie algebras Let V be a vector space with a product $[\ ,\ ] : V \times V \to V$ that satisfies the conditions

$$\text{Antisymmetry:} \qquad [v, w] + [w, v] = 0 , \tag{2}$$

$$\text{Jacob identity:} \qquad [v, [w, u]] + [u, [v, w]] + [w, [u, v]] = 0 , \tag{3}$$

for any $w, v \in V$. Then $(V, [\ ,\ ])$ is a *Lie algebra*. Evaluation of the product $[\ ,\ ]$ on a basis $\{T_a\}$ of V (the *generators*) gives rise to the *structure constants* $f_{ab}^{\ \ c}$ via

$$[T_a, T_b] = f_{ab}^{\ \ c} T_c . \tag{4}$$

a) Consider a three-dimensional vector space V with the basis elements T_1, T_2, T_3. Define an antisymmetric product $[\ ,\ ]$ as follows:

$$[T_1, T_2] = -[T_2, T_1] = iT_2 \tag{5a}$$

$$[T_2, T_3] = -[T_3, T_2] = iT_3 \tag{5b}$$

$$[T_1, T_3] = -[T_3, T_1] = iT_1 \tag{5c}$$

Does this define a Lie algebra?

b) What relations do, respectively, the antisymmetry and the Jacobi identity impose on the structure constants $f_{ab}^{\ \ c}$ of a Lie algebra?

c) The exponential map allows to map a Lie algebra element X into an group element $\exp(X)$ of the corresponding Lie group. Show that

$$\det(\exp(X)) = \exp(\mathrm{Tr}(X)) , \tag{6}$$

for any matrix $X \in \mathrm{Mat}(n, \mathbb{C})$. (Hint: proceed in three steps: diagonal X, diagonalisable X, and generic X.)

d) Determine the properties of Lie algebra elements for the Lie groups

$$SU(n) = \left\{ U \in \mathrm{Mat}(n,\mathbb{C}) \,\middle|\, UU^\dagger = U^\dagger U = 1 \,, \det(U) = 1 \right\} \,, \tag{7a}$$

$$SO(n) = \left\{ O \in \mathrm{Mat}(n,\mathbb{R}) \,\middle|\, OO^T = O^T O = 1 \,, \det(O) = 1 \right\} \,, \tag{7b}$$

by employing (6) and $\exp(X) \approx 1 + X + \dots$.

Problem 3: Gauge Transformations Consider a scalar field ϕ and a gauge field A_μ subject to the (non-abelian) gauge transformations

$$\phi \mapsto \phi^U \coloneqq U\Phi \,, \qquad A_\mu \mapsto A_\mu^U \coloneqq U A_\mu U^{-1} - \frac{i}{g}(\partial_\mu U)U^{-1} \,, \tag{8}$$

for U a smooth map on space-time taking values in a representation of a Lie group G.

a) Show that the transformations (8) form a group.

b) Show that the covariant derivative $D_\mu \phi \coloneqq (\partial_\mu - igA_\mu)\phi$ transforms covariantly under (8), i.e. $(D_\mu\phi)^U = U(D_\mu\phi)$.

c) Derive the explicit expression of the field strength F, which is defined as

$$F_{\mu\nu}\phi \coloneqq \frac{i}{g}\left[D_\mu, D_\nu\right]\phi \,, \qquad \text{for any } \phi \,. \tag{9}$$

Derive the transformation behaviour under (8). What happens if G is abelian?

d) Derive the infinitesimal versions of (8) by writing

$$U = \exp\left(i \sum_A \omega_A T^A \right) \,, \tag{10}$$

wherein ω_A are *gauge parameters*, i.e. smooths functions on the space-time. (N.B. the sum over A extends from 1 to the rank of the corresponding gauge group G.) Here, T^A are representation matrices for the generators of the Lie algebra.

e) (Bonus) Rewrite the infinitesimal gauge transformation of A_μ by using the covariant derivative of the adjoint representation.